Guarded Cubical Type Theory: Path Equality for Guarded Recursion

Technical appendix

Lars Birkedal Aleš Bizjak Ranald Clouston Hans Bugge Grathwohl
Bas Spitters Andrea Vezzosi

Contents

1 Guarded Cubical Type Theory **2**

2 Denotational semantics **2**

2.1 The language $\mathcal{L}$ . . . 2
2.1.1 Assumption 1: The interval type . . . 3
2.1.2 Definable concepts . . . 4
2.1.3 Assumption 2: Glueing . . . 7
2.1.4 Assumption 3: Fibrant universe . . . 7
2.1.5 Assumption 4: $\forall$ . . . 7
2.2 A model of CTT . . . 7
2.2.1 Interpreting dependent function types . . . 8
2.2.2 Interpreting dependent sum types . . . 8
2.2.3 Interpreting base types . . . 8
2.2.4 Interpreting systems . . . 10
2.2.5 Interpreting path types . . . 10
2.2.6 Interpreting glue types . . . 11
2.2.7 Interpreting the universe . . . 11
2.3 A concrete model of $\mathcal{L}$ . . . 11
2.3.1 Assumption 1 is satisfied . . . 11
2.3.2 Assumption 2 is satisfied . . . 12
2.3.3 Assumption 3 is satisfied . . . 14
2.3.4 Assumption 4 is satisfied . . . 14
2.4 More models of $\mathcal{L}$ . . . 15
2.4.1 Assumption 1 is satisfied . . . 16
2.4.2 Assumption 2 is satisfied . . . 16
2.4.3 Assumption 3 is satisfied . . . 16
2.4.4 Assumption 4 is satisfied . . . 19
2.5 A model of GCTT . . . 19
2.5.1 The functor $\triangleright$ . . . 19
2.5.2 Interpreting later types . . . 23
2.6 Summary of the semantics of GCTT . . . 25

Delayed substitutions, $\vdash \xi : \Gamma \rightarrowtriangle \Gamma'$

$$\frac{\Gamma \vdash}{\vdash \cdot : \Gamma \rightarrowtriangle \cdot} \qquad \frac{\vdash \xi : \Gamma \rightarrowtriangle \Gamma' \qquad \Gamma \vdash t : \triangleright \xi.A}{\vdash \xi\,[x \leftarrow t] : \Gamma \rightarrowtriangle \Gamma', x : A}$$

Well-formed types, $\Gamma \vdash A$

$$\frac{\Gamma, \Gamma' \vdash A \qquad \vdash \xi : \Gamma \rightarrowtriangle \Gamma'}{\Gamma \vdash \triangleright \xi.A}$$

Well-typed terms, $\Gamma \vdash t : A$

$$\frac{\Gamma, \Gamma' \vdash A : \mathsf{U} \qquad \vdash \xi : \Gamma \rightarrowtriangle \Gamma'}{\Gamma \vdash \triangleright \xi.A : \mathsf{U}} \qquad \frac{\Gamma, \Gamma' \vdash t : A \qquad \vdash \xi : \Gamma \rightarrowtriangle \Gamma'}{\Gamma \vdash \mathsf{next}\,\xi.t : \triangleright \xi.A} \qquad \frac{\Gamma \vdash r : \mathbb{I} \qquad \Gamma, x : \triangleright A \vdash t : A}{\Gamma \vdash \mathsf{dfix}^r\, x.t : \triangleright A}$$

Figure 1: Overview of new rules in GCTT (part 1).

1 Guarded Cubical Type Theory

We define *guarded cubical type theory* (GCTT) to be an extension of cubical type theory (CTT)[1] [3] with the following syntax:

$$\begin{array}{lcl} t, u, A, B & ::= & \cdots \mid \mathsf{next}\,\xi.t \mid \triangleright \xi.A \mid \mathsf{dfix}^r\, x.t \\ \xi & ::= & \cdot \mid \xi\,[x \leftarrow t] \end{array}$$

along with the typing rules of Figure 1 and Figure 2.

2 Denotational semantics

In this section we provide the necessary semantic constructions that can be used to interpret the type theory GCTT.

2.1 The language $\mathcal{L}$

Instead of formulating our model directly using regular mathematics, we will specify a type-theoretic language $\mathcal{L}$, tailor-made for the purpose of our model. It is based on the internal logic of the presheaf topos of cubical sets, $\mathrm{Set}^{\mathcal{C}}$.

$\mathcal{L}$ is an extension of W. Phoa's [9, Appendix I] *dependent predicate logic*; see also [6, D4.3,4.4]. Figure 3 contains an overview of the types of judgements. Note that a *proposition* is a term of type Ω. Formally the logical inference judgements will be of the form $\Gamma \vdash \varphi = \mathrm{true} : \Omega$, but in practice we will stick to a more informal notation, e.g., writing Γ, φ for a context where φ holds, instead of $\Gamma, p : \mathrm{Eq}(\varphi, \mathrm{true})$. In addition to the equality proposition $\mathrm{Eq}(t, u) : \Omega$, we also have an extensional identity type $\mathrm{Id}_A(t, u)$ with equality reflection:

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash t, u : A}{\Gamma \vdash \mathrm{Id}_A(t, y)} \qquad \frac{\Gamma \vdash t = u : A}{\Gamma \vdash \mathrm{refl} : \mathrm{Id}_A(t, u)} \qquad \frac{\Gamma \vdash p : \mathrm{Id}_A(t, u)}{\Gamma \vdash t = u : A}$$

Id (the type) and Eq (the proposition) are equally expressive, but for presentation purposes it is practical to have both: Using Id we can easily express the type of *partial elements* without reference to Ω, e.g., an

[1] An overview of the rules of CTT can be found at http://www.cse.chalmers.se/~coquand/selfcontained.pdf.

Type equality, $\Gamma \vdash A = B$ (Congruence and equivalence rules are omitted)

$$\frac{\vdash \xi\,[x \leftarrow t] : \Gamma \twoheadrightarrow \Gamma', x : B \qquad \Gamma, \Gamma' \vdash A}{\Gamma \vdash \triangleright\xi\,[x \leftarrow t]\,.A = \triangleright\xi.A}$$

$$\frac{\vdash \xi\,[x \leftarrow t, y \leftarrow u]\,\xi' : \Gamma \twoheadrightarrow \Gamma', x : B, y : C, \Gamma'' \qquad \Gamma, \Gamma' \vdash C \qquad \Gamma, \Gamma', x : B, y : C, \Gamma'' \vdash A}{\Gamma \vdash \triangleright\xi\,[x \leftarrow t, y \leftarrow u]\,\xi'.A = \triangleright\xi\,[y \leftarrow u, x \leftarrow t]\,\xi'.A}$$

$$\frac{\vdash \xi : \Gamma \twoheadrightarrow \Gamma' \qquad \Gamma, \Gamma', x : B \vdash A \qquad \Gamma, \Gamma' \vdash t : B}{\Gamma \vdash \triangleright\xi\,[x \leftarrow \mathsf{next}\,\xi.t]\,.A = \triangleright\xi.A[t/x]}$$

Term equality, $\Gamma \vdash t = u : A$ (Congruence and equivalence rules are omitted)

$$\frac{\vdash \xi\,[x \leftarrow t] : \Gamma \twoheadrightarrow \Gamma', x : B \qquad \Gamma, \Gamma' \vdash A : \mathsf{U}}{\Gamma \vdash \triangleright\xi\,[x \leftarrow t]\,.A = \triangleright\xi.A : \mathsf{U}}$$

$$\frac{\vdash \xi\,[x \leftarrow t, y \leftarrow u]\,\xi' : \Gamma \twoheadrightarrow \Gamma', x : B, y : C, \Gamma'' \qquad \Gamma, \Gamma' \vdash C : \mathsf{U} \qquad \Gamma, \Gamma', x : B, y : C, \Gamma'' \vdash A : \mathsf{U}}{\Gamma \vdash \triangleright\xi\,[x \leftarrow t, y \leftarrow u]\,\xi'.A = \triangleright\xi\,[y \leftarrow u, x \leftarrow t]\,\xi'.A : \mathsf{U}}$$

$$\frac{\vdash \xi : \Gamma \twoheadrightarrow \Gamma' \qquad \Gamma, \Gamma', x : B \vdash A : \mathsf{U} \qquad \Gamma, \Gamma' \vdash t : B}{\Gamma \vdash \triangleright\xi\,[x \leftarrow \mathsf{next}\,\xi.t]\,.A = \triangleright\xi.A[t/x] : \mathsf{U}} \qquad \frac{\vdash \xi\,[x \leftarrow t] : \Gamma \twoheadrightarrow \Gamma', x : B \qquad \Gamma, \Gamma' \vdash u : A}{\Gamma \vdash \mathsf{next}\,\xi\,[x \leftarrow t]\,.u = \mathsf{next}\,\xi.u : \triangleright\xi.A}$$

$$\frac{\vdash \xi\,[x \leftarrow t, y \leftarrow u]\,\xi' : \Gamma \twoheadrightarrow \Gamma', x : B, y : C, \Gamma'' \qquad \Gamma, \Gamma' \vdash C \qquad \Gamma, \Gamma', x : B, y : C, \Gamma'' \vdash v : A}{\Gamma \vdash \mathsf{next}\,\xi\,[x \leftarrow t, y \leftarrow u]\,\xi'.v = \mathsf{next}\,\xi\,[y \leftarrow u, x \leftarrow t]\,\xi'.v : \triangleright\xi\,[x \leftarrow t, y \leftarrow u]\,\xi'.A}$$

$$\frac{\vdash \xi : \Gamma \twoheadrightarrow \Gamma' \qquad \Gamma, \Gamma', x : B \vdash u : A \qquad \Gamma, \Gamma' \vdash t : B}{\Gamma \vdash \mathsf{next}\,\xi\,[x \leftarrow \mathsf{next}\,\xi.t]\,.u = \mathsf{next}\,\xi.u[t/x] : \triangleright\xi.A[t/x]} \qquad \frac{\Gamma \vdash t : \triangleright\xi.A}{\Gamma \vdash \mathsf{next}\,\xi\,[x \leftarrow t]\,.x = t : \triangleright\xi.A}$$

$$\frac{\Gamma, x : \triangleright A \vdash t : A}{\Gamma \vdash \mathsf{dfix}^1\,x.t = \mathsf{next}\,t[\mathsf{dfix}^0\,x.t/x] : \triangleright A}$$

Figure 2: Overview of new rules in GCTT (part 2).

element of B only defined when $t = u$: $\Gamma \vdash b : \mathrm{Id}_A(t, u) \to B$. Such terms, however, are unwieldy to work with since you need to carry around an explicit equality proof (which will be equal to refl anyway). Therefore we will implicitly convert back and forth between the type theoretic and the logical representation, which for our previous example means that in a context where $t = u$ we will write $b : B$.

We also assume that $\mathcal{L}$ contains a universe $\mathcal{U}$ of small types, along with the "elements-of" functor El.

2.1.1 Assumption 1: The interval type

In $\mathcal{L}$ we have a type $\mathbb{I}$ with

$$0, 1 : \mathbb{I} \qquad \wedge, \vee : \mathbb{I} \to \mathbb{I} \to \mathbb{I} \qquad 1 - \cdot : \mathbb{I} \to \mathbb{I}$$

$\Gamma \vdash$	well-formed context
$\Gamma \vdash A$	well-formed type
$\Gamma \vdash t : A$	typing judgement
$\Gamma \vdash A = B$	type equality
$\Gamma \vdash t = u : A$	term equality

Figure 3: Judgements of $\mathcal{L}$

which is a *De Morgan algebra*, i.e., $(\mathbb{I}, 0, 1, \wedge, \vee)$ is a bounded distributive lattice, and $1 - \cdot$ is an involution which satisfies De Morgan's laws:

$$1 - (i \wedge j) = (1 - i) \vee (1 - j),$$
$$1 - (i \vee j) = (1 - i) \wedge (1 - j).$$

In addition to the De Morgan algebra laws we assume the following two axioms

$$0 \neq 1$$
$$i \vee j = 1 \implies i = 1 \vee j = 1$$

the latter of which we refer to as the *disjunction property*.

2.1.2 Definable concepts

We can now define some useful abbreviations in $\mathcal{L}$.

Faces. Using the interval we define the type $\mathbb{F}$ as the image of the function $\cdot = 1 : \mathbb{I} \to \Omega$, where Ω is the subobject classifier. More precisely, $\mathbb{F}$ is the subset type

$$\mathbb{F} \triangleq \{p : \Omega \mid \exists (i : \mathbb{I}), p = (i = 1)\}$$

We will implicitly use the inclusion $\mathbb{F} \to \Omega$. The following lemma in particular states that the inclusion is compatible with all the lattice operations, hence omitting it is justified.

Lemma 2.1.

- $\mathbb{F}$ *is a lattice for operations inherited from* Ω.
- *The corestriction* $\cdot = 1 : \mathbb{I} \to \mathbb{F}$ *is a lattice homomorphism. It is not injective.*
- $\mathbb{F}$ *inherits the disjunction property from* $\mathbb{I}$.
- *The operation* $\varphi = 1 \mapsto \varphi = 0$ *does* not *make* $\mathbb{F}$ *into a DM-algebra:* *For all* i*,* $(i = 1) \wedge ((1 - i) = 1) =_\Omega \bot$*. However, if* $((1 - i) = 1) \vee (i = 1) =_\Omega \top$*, then* $i = 0$ *or* $i = 1$*.*

Given a proposition $\Gamma \vdash \varphi : \mathbb{F}$ we define the type

$$[\varphi] \triangleq \mathrm{Id}_{\mathbb{F}}(\varphi, \top).$$

Remark 2.2. Note that we have the following logical equivalence

$$\Gamma \mid \cdot \vdash (\exists! p : [\varphi], \top) \iff \varphi.$$

Given $\Gamma \vdash A$ and $\Gamma \vdash \varphi : \mathbb{F}$ we say that a term t is a *partial element* of A of extent φ, if $\Gamma \vdash t : \Pi(p : [\varphi]).A$. If we are in a context with $p : [\varphi]$, then we will treat such a partial element t as a term of type A, leaving implicit the application to the proof p, i.e., we will treat t as $t\,p$. We will often write $\Gamma, [\varphi]$ instead of $\Gamma, p : [\varphi]$ when we do not mention the proof term p explicitly. Given $\Gamma, p : [\varphi] \vdash B$ we write B^φ for the dependent function space $\Pi(p : [\varphi]).B$ and leave the proof p implicit.

If we have a term $\Gamma, p : [\varphi] \vdash u : A$ (a partial element), then we can define

$$A[\varphi \mapsto u] \triangleq \Sigma(a : A).\,(\mathrm{Id}_A(a, u))^\varphi\,.$$

Systems. Given $\Gamma \vdash A$, assume we have the following:

$$\begin{aligned}
\Gamma &\vdash \varphi_1, \dots, \varphi_n : \mathbb{F} \\
\Gamma &\vdash \varphi_1 \vee \cdots \vee \varphi_n = \top \\
\Gamma, [\varphi_1] &\vdash t_1 : A \\
&\vdots \\
\Gamma, [\varphi_n] &\vdash t_n : A \\
\Gamma, [\varphi_i \wedge \varphi_j] &\vdash t_i = t_j : A, \quad \text{for all } i, j.
\end{aligned}$$

In other words: We have n partial elements of A which agree with each other. We can use the *axiom of definite description* to define a term

$$[\varphi_1 t_1, \dots, \varphi_n t_n] \triangleq \text{the } x^A \text{ such that } \chi(x)$$

where

$$\chi(x) \triangleq (\varphi_1 \wedge (x = t_1)) \vee \cdots \vee (\varphi_n \wedge (x = t_n)).$$

We call this term a *system*. The condition for using definite description is a proof (in the logic) of *unique existence* of such a term. This follows almost directly from the assumptions and Remark 2.2.

Using systems, we generalise an earlier definition: We define

$$A[\varphi_1 \mapsto t_1, \dots, \varphi_n \mapsto t_n] \triangleq A[\varphi_1 \vee \cdots \vee \varphi_n \mapsto [\varphi_1 t_1, \dots, \varphi_n t_n]],$$

where the type on the right hand side is using the earlier definition. Note that $A[\varphi \mapsto t]$ is unambiguous, as we have $\Gamma, [\varphi] \vdash [\varphi t] = t : A$.

Compositions. Given $\Gamma \vdash A$, we can define the type of *compositions*:

$$\begin{aligned}
\Phi(\Gamma; A) \triangleq \Pi(&\gamma : \mathbb{I} \to \Gamma) \\
&(\varphi : \mathbb{F}) \\
&(u : \Pi(i : \mathbb{I}).\,[\varphi] \to A(\gamma(i))). \\
&A(\gamma(0))[\varphi \mapsto u(0)] \to A(\gamma(1))[\varphi \mapsto u(1)].
\end{aligned}$$

We say that a type $\Gamma \vdash A$ is *fibrant* if there is a term $\vdash \mathbf{c} : \Phi(\Gamma; A)$ (A *has compositions*).

Fillings. Given $\Gamma \vdash A$, we can define the type of *(Kan) fillings*:

$$\begin{aligned}
\Psi(\Gamma, A) \triangleq \Pi(&\gamma : \mathbb{I} \to \Gamma) \\
&(\varphi : \mathbb{F}) \\
&(u : \Pi(i : \mathbb{I}).\,[\varphi] \to A(\gamma(i))) \\
&(a_0 : A(\gamma(0))[\varphi \mapsto u(0)]) \\
&(i : \mathbb{I}). \\
&A(\gamma(i))[\varphi \mapsto u(i), (1 - i) \mapsto \pi_1 a_0].
\end{aligned}$$

If we have a filling operation $\mathbf{f} : \Psi(\Gamma, A)$ then we can get a *path lifting* operation

$$\begin{aligned}\boldsymbol{\ell} : \Pi(&\gamma : \mathbb{I} \to \Gamma)\\ &(a_0 : A(\gamma(0)))\\ &(i : \mathbb{I}).\\ &A(\gamma(i))[(1-i) \mapsto a_0],\end{aligned}$$

by taking the simple case of $\mathbf{f}$ where φ is $\bot$, and u therefore is uniquely determined (since it is a partial function defined where $\bot$ holds).

Fillings are special cases of compositions.

Lemma 2.3 (Fillings from compositions)**.** *If we have a fibrant type $\Gamma \vdash A$ with $\mathbf{c}_A : \Phi(\Gamma; A)$, then we have a filling operation $\vdash \mathbf{f} : \Psi(\Gamma, A)$.*

Proof. We introduce the variables of the proper types:

$$\begin{aligned}&\gamma : \mathbb{I} \to \Gamma,\\ &\varphi : \mathbb{F},\\ &u : \Pi(i : \mathbb{I}).[\varphi] \to A(\gamma(i)),\\ &a_0 : A(\gamma(0))[\varphi \mapsto u(0)],\\ &i : \mathbb{I}.\end{aligned}$$

We need to find a term of type

$$A(\gamma(i))[\varphi \mapsto u(i), (i=0) \mapsto \pi_1 a_o].$$

We check that the following system is well-defined (in a context with $\varphi \vee (i=0)$):

$$[\varphi u(i \wedge j), (i=0)\pi_1 a_0].$$

- If φ, then $u(i \wedge j) : A(\gamma(i \wedge j))$.
- If $i = 0$, then $\pi_1 a_0 : A(\gamma(0)) = A(\gamma(i \wedge j))$.
- If φ and $i = 0$, then $\pi_1 a_0 = u(0) = u(i \wedge j)$.

Note also that this means that

$$A(\gamma(0))[\varphi \mapsto u(0)] = A(\gamma(0))[\varphi \mapsto u(0), (i=0) \mapsto \pi_1 a_0],$$

and therefore we can write the following term:

$$\mathbf{c}_A\ (\lambda j.\gamma(i \wedge j))\ (\varphi \vee (i=0))\ (\lambda j.[\varphi u(i \wedge j), (i=0)\pi_1 a_0])\ a_0$$

which has the type

$$A(\gamma(i))[\varphi \mapsto u(i), (i=0) \mapsto \pi_1 a_o],$$

as was needed. □

Path types. Given $\Gamma \vdash A$ and terms $\Gamma \vdash t, u : A$, we can define the *Path type*

$$\text{Path}_A\ t\ u \triangleq \Pi(i : \mathbb{I}).A[(1-i) \mapsto t, i \mapsto u].$$

2.1.3 Assumption 2: Glueing

There is a type for *glueing* with the following type formation and typing rules

$$\frac{\Gamma \vdash A \qquad \Gamma, [\varphi] \vdash T \qquad \Gamma, [\varphi] \vdash f : T \to A}{\Gamma \vdash \text{Glue } [\varphi \mapsto (T, f)] \ A} \qquad \frac{\Gamma \vdash b : \text{Glue } [\varphi \mapsto (T, f)] \ A}{\Gamma \vdash \text{unglue}\, b : A[\varphi \mapsto f\, b]}$$

$$\frac{\Gamma, [\varphi] \vdash f : T \to A \qquad \Gamma, [\varphi] \vdash t : T \qquad \Gamma \vdash a : A[\varphi \mapsto f\, t]}{\Gamma \vdash \text{glue } [\varphi \mapsto t] \ a : \text{Glue } [\varphi \mapsto (T, f)] \ A}$$

Additionally we have the following equations for glueing:

$$\begin{aligned} \text{glue } [1 \mapsto t] \ a &= t, \\ \text{glue } [\varphi \mapsto b] \ (\text{unglue}\, b) &= b, \\ \text{unglue}(\text{glue } [\varphi \mapsto t] \ a) &= a. \end{aligned}$$

2.1.4 Assumption 3: Fibrant universe

There is a *fibrant universe* $\mathcal{U}_f$ which contains all of the codes in $\mathcal{U}$ for which there is an associated composition operator:

$$\frac{\Gamma \vdash a : \mathcal{U} \qquad \vdash \mathbf{c} : \Phi(\Gamma; \text{El}(a))}{\Gamma \vdash (\!| a, \mathbf{c} |\!) : \mathcal{U}_f} \qquad \frac{\Gamma \vdash a : \mathcal{U}_f}{\Gamma \vdash \text{El}(a)} \qquad \frac{\Gamma \vdash a : \mathcal{U}_f}{\vdash \text{Comp}(a) : \Phi(\Gamma; \text{El}(a))}$$

2.1.5 Assumption 4: $\forall$

Finally we assume that the map $\varphi \mapsto \lambda\_.\varphi : \mathbb{F} \to (\mathbb{I} \to \mathbb{F})$ has an internal right adjoint $\forall$. By this we mean the following correspondence for any $\varphi : \mathbb{F}$ and any $f : \mathbb{I} \to \mathbb{F}$.

$$(\varphi \Rightarrow \forall (i : \mathbb{I}), f(i)) \iff (\varphi \Rightarrow \forall (f))\,.$$

2.2 A model of CTT

We define a *category with families* [5] by specifying the type and term functors. The idea is to reuse the types and terms of the language $\mathcal{L}$, but we only take the *fibrant* types, i.e., the ones with associated composition operators.

$$\text{Ty}(\Gamma) \triangleq \left\{ ([A], [\mathbf{c}_A]) \;\middle|\; \begin{array}{l} \Gamma \vdash A \\ \vdash \mathbf{c}_A : \Phi(\Gamma; A) \end{array} \right\}$$

$$\text{Tm}(\Gamma, ([A], [\mathbf{c}_A])) \triangleq \{[t] \mid \Gamma \vdash t : A\}\,.$$

where we use $[A]$ and $[t]$ respectively for the equivalence classes of A and t modulo judgemental equality of $\mathcal{L}$. In the following we will omit the mention of equivalence classes and work with representatives. This is justified since all the operations in $\mathcal{L}$ respect judgemental equality.

Note, that the context Γ need not be fibrant. Context extension and projections can just be taken directly from the internal language: $\Gamma.A \triangleq \Sigma \Gamma A$, $\mathsf{p} \triangleq \pi_1, \mathsf{q} \triangleq \pi_2$.

When we interpret CTT we need to find both a type and a composition operator in $\mathcal{L}$ for each type in CTT.

2.2.1 Interpreting dependent function types

Assume that $[\![\Gamma \vdash A']\!] = (A, \mathbf{c}_A)$ and $[\![\Gamma, x : A' \vdash B']\!] = (B, \mathbf{c}_B)$. We define

$$[\![\Gamma \vdash (x : A') \to B']\!] \triangleq (\Pi(x : A).B, \mathbf{c})$$

where $\mathbf{c} : \Phi(\Gamma; \Pi(x : A).B)$ comes from the following lemma:

Lemma 2.4. *Π-types preserve compositions. I.e., if we have composition terms $\mathbf{c}_A : \Phi(\Gamma; A)$ and $\mathbf{c}_B : \Phi(\Gamma.A; B)$, then we can form a new composition $\mathbf{c}_{\Pi(x:A).B} : \Phi(\Gamma, \Pi(x : A).B)$.*

Proof. Recall that Π-types commutes with substitution:

$$(\Pi(x : A).B)(\gamma) = \Pi(x : A(\gamma)).B(\gamma),$$

where $B(\gamma)$ is a type in the context with A. We introduce the variables:

$$\begin{aligned}
&\gamma : \mathbb{I} \to \Gamma,\\
&\varphi : \mathbb{F},\\
&u : \Pi(i : \mathbb{I}).[\varphi] \to \Pi(a : A(\gamma(i))).B(\gamma(i)),\\
&c_0 : (\Pi(a : A(\gamma(0))).B(\gamma(0)))[\varphi \mapsto u(0)].
\end{aligned}$$

We need to find an element in

$$\Pi(a : A(\gamma(1))).B(\gamma(1)),$$

along with a proof that it is $u(1)$ when $\varphi = 1$.

Let $a_1 : A(\gamma(1))$ be given. We define $a(i) : A(\gamma(i))[i \mapsto a_1]$ by using path lifting on a_1, i.e.,

$$a(i) \triangleq \boldsymbol{\ell}\ (\lambda i.\gamma(1 - i))\ a_1\ (1 - i).$$

Then

$$b_1 \triangleq \mathbf{c}_B\ (\lambda i.\ \langle \gamma(i), a(i) \rangle)\ \varphi\ (\lambda i.u(i)(a(i)))$$

will have the type $B(\gamma(1))[\varphi \mapsto u(1)a_1]$. So $\lambda a_1.\pi_1 b_1$ has the type we are looking for. Now assume $\varphi = \top$; then $\lambda a_1.b_1 = \lambda a_1.u(i)a_1 = u(i)$, which is what we needed. ☐

The above proof is analogous to the equality judgement for compositions at Π-types in CTT [3].

2.2.2 Interpreting dependent sum types

Dependent sum types $(x : A) \times B$ are interpreted by Σ-types from $\mathcal{L}$, along with the composition operation that comes from the following lemma:

Lemma 2.5. *Σ-types preserve compositions. I.e., if we have composition terms $\mathbf{c}_A : \Phi(\Gamma; A)$ and $\mathbf{c}_B : \Phi(\Gamma.A; B)$, then we can form a new composition $\mathbf{c}_{\Sigma(x:A).B} : \Phi(\Gamma, \Sigma(x : A).B)$.*

This proof is analogous to the equality judgement for compositions at Σ-types in CTT [3].

2.2.3 Interpreting base types

If a type A is independent of $\mathbb{I}$, then we say it is *discrete*. Externally, this means that it is a constant presheaf, i.e., $A = \Delta(A')$ for some $A' \in \mathrm{Set}$, where $\Delta : \mathrm{Set} \to \mathrm{Set}^{\mathcal{C}}$ is the constant presheaf functor. Internally, it means that the following type is inhabited

$$\Pi(i, j : \mathbb{I})(a : \mathbb{I} \to A).a(i) = a(j).$$

Externally we have an isomorphism $(\Delta(A))^{\mathbb{I}} \cong \Delta(A)$, so if a type is discrete in the external sense, then it will also be discrete in the internal sense.

Lemma 2.6. *For any cubical set A and any $I \in \mathcal{C}$ and $i \notin I$ the function $\beta^i_I : A^{\mathbb{I}}(I) \to A(I, i)$ defined as*

$$\beta^i_I(f) = f_\iota(i),$$

where $\iota : I \to I, i$ is the inclusion, is an isomorphism. Moreover the family β is natural in I and i in the following sense. For any $J \in \mathcal{C}$ and $j \notin J$ and any $g : I \to J$ we have

$$A(g + (i \mapsto j)) \circ \beta^i_I = \beta^j_J \circ A^{\mathbb{I}}(g).$$

Corollary 2.7. *If the obvious morphism $A \to A^{\mathbb{I}}$ is an isomorphism, then A is isomorphic to an object of the form $\Delta(a)$ for some $a \in \mathrm{Set}$.*

Proof. The obvious morphism is of course the constant map $a \mapsto \lambda\_.a$. Using Lemma 2.6 we thus have that for each I and $i \notin I$, $A(\iota) : A(I) \to A(I, i)$ is an isomorphism, where, again, ι is the inclusion. From this we have that for all I, the inclusion $A(\iota_I) : A(\emptyset) \to A(I)$ is an isomorphism.

Define $a = A(\emptyset)$ and $\alpha : \Delta(a) \to A$ as

$$\alpha_I = A(\iota_I).$$

We then have for any $f : I \to J$ the following

$$A(f) \circ \alpha_I = A(f \circ \iota_I) = A(\iota_J).$$

The latter because $f \circ \iota_I$ and ι_J are both maps from the empty set, hence they are equal.

By the previous lemma each α_I is an isomorphism and by the preceding calculation α is a natural transformation. Hence α is a natural isomorphism. □

Lemma 2.8. *If A is isomorphic to $\Delta(a)$ for some $a \in \mathrm{Set}$ then the obvious morphism $A \to A^{\mathbb{I}}$ is an isomorphism.*

Proof. The inverse to the isomorphism β in Lemma 2.6 is the morphism α^i_I

$$\alpha^i_I(a)_f(j) = A([f, (i \mapsto j)])(a).$$

By assumption $A(\iota)$ for any inclusion $\iota : I \to I, i$ is an isomorphism. It is easy to compute that the canonical morphism $A \to A^{\mathbb{I}}$ arises as the composition of $A(\iota)$ and α^i_I. □

Proposition 2.9. *Let A be a cubical set. The formula*

$$i : \mathbb{I}, j : \mathbb{I}, f : (\mathbb{I} \to A) \mid \cdot \vdash f(i) = f(j)$$

holds in the internal language if and only if A is isomorphic to $\Delta(a)$ for some $a \in \mathrm{Set}$.

Proof. Suppose the formula holds. Then it is easy to see that the constant map from A to $A^{\mathbb{I}}$ is an isomorphism (the inverse is given, for instance, by evaluation at 0). Corollary 2.7 implies the result.

Conversely assume $A \cong \Delta(a)$ for some $a \in \mathrm{Set}$. Then by Lemma 2.8 the canonical map $\mathrm{const} : A \to A^{\mathbb{I}}$ is an isomorphism. Hence it is internally surjective. Thus for any $f : \mathbb{I} \to A$ there is an a in A, such that $\mathrm{const}\, a = f$. From this we immediately have $f(i) = f(j)$ for any i and j in $\mathbb{I}$. □

Lemma 2.10. *Every discrete type $\vdash A$ is fibrant, i.e., it has a composition operator $\mathbf{c}_A : \Phi(\cdot; A)$.*

Proof. Since A is discrete, we have that $u(0) = u(1)$ for any $u : \Pi(i : \mathbb{I}).[\varphi] \to A$. Therefore $A[\varphi \mapsto u(0)] = A[\varphi \mapsto u(1)]$, so we can choose the constant function $\lambda\gamma, \varphi, u, a.a$ to be $\mathbf{c}_A$, since this will be of type $\Phi(\cdot, A)$. □

If we have a composition operator $\mathbf{c}_A : \Phi(\cdot; A)$ then we can always construct a weakened version $\mathbf{c}'_A : \Phi(\Gamma; A)$ for any Γ, since A does not depend on Γ.

Therefore we can interpret the natural number type:

$$[\![\Gamma \vdash \mathsf{N}]\!] \triangleq (\mathbb{N}, \mathbf{c}_{\mathbb{N}}),$$

where $\mathbf{c}_{\mathbb{N}}$ is the composition that we get from Lemma 2.10.

2.2.4 Interpreting systems

We interpret the systems of CTT by using the systems of $\mathcal{L}$, and by using the fact that systems preserve compositions: If we have a system $\Gamma \vdash [\varphi_1 A_1, \ldots, \varphi_n A_n]$, then we can define a new composition using a system consisting of the compositions of all the components:

$$\mathbf{c} \triangleq \lambda\gamma, \psi, u, a_0.[\varphi_1(\gamma\, i)(\mathbf{c}_{A_1}\, \gamma_1\, \psi\, u\, a_0), \ldots, \varphi_n(\gamma\, i)(\mathbf{c}_{A_n}\, \gamma_n\, \psi\, u\, a_0)] \;:\; \Phi(\Gamma; [\varphi_1 A_1, \ldots, \varphi_n A_n]),$$

where $\gamma_m : \mathbb{I} \to \Gamma, [\varphi_m]$ is the context map γ extended with the witness of $[\varphi_m]$.

2.2.5 Interpreting path types

We interpret the path types:

$$[\![\Gamma \vdash \mathsf{Path}\, A\ t\ s]\!] \triangleq (\mathrm{Path}_{A'}\, [\![t]\!]\ [\![s]\!], \mathbf{c}),$$

where $[\![A]\!] = (A', \mathbf{c}_A)$ and $\mathbf{c} : \Phi(\Gamma; \mathrm{Path}_{A'}\, [\![t]\!]\ [\![s]\!])$ comes from Lemma 2.11.

Lemma 2.11. *Path-types preserve composition, i.e., if $\Gamma \vdash A$ is fibrant, then for any $\Gamma \vdash t, s : A$, we will have a composition operator $\mathbf{c} : \Phi(\Gamma; \mathrm{Path}_A\ t\ s)$.*

Proof. First note that if we have $\Gamma \vdash \mathrm{Path}_A\ t\ s :$ and $\vdash \gamma : \Gamma$, then

$$(\mathrm{Path}_A\ t\ s)(\gamma) = \mathrm{Path}_{A(\gamma)}\ t(\gamma)\ s(\gamma) = \Pi(i : \mathbb{I}).A(\gamma) \begin{bmatrix} i = 0 \mapsto t(\gamma) \\ i = 1 \mapsto s(\gamma) \end{bmatrix}.$$

Now let

$$\begin{aligned}
&\gamma : \mathbb{I} \to \Gamma \\
&\varphi : \mathbb{I} \\
&u : \Pi(j : \mathbb{I}).[\varphi] \to \mathrm{Path}_{A(\gamma j)}\ t(\gamma j)\ s(\gamma j) \\
&p_0 : (\mathrm{Path}_{A(\gamma 0)}\ t(\gamma 0)\ s(\gamma 0))[\varphi \mapsto u0]
\end{aligned}$$

be given. Our goal is to find a term p_1 such that

$$p_1 : (\mathrm{Path}_{A(\gamma 1)}\ t(\gamma 1)\ s(\gamma 1))[\varphi \mapsto u1].$$

We will do this by finding a term $q : \Pi(i : \mathbb{I}).A(\gamma 1)[\varphi \mapsto u1i]$, for which we verify that $q0 = t(\gamma 1)$ and $q1 = s(\gamma 1)$, in other words,

$$q : \Pi(i : \mathbb{I}).A(\gamma 1)[\varphi \mapsto u1i, (1-i) \mapsto t(\gamma 1), i \mapsto s(\gamma 1)]$$

as this will be equivalent to having such a p_1.

Let $i : \mathbb{I}$. By leaving some equality proofs implicit we can define the system

$$r(j) \triangleq [\varphi uji, (1-i)t(\gamma j), is(\gamma j)] : \Pi(j : \mathbb{I}).[\varphi \vee (1-i) \vee i] \to A(\gamma j),$$

which is well-defined because $uj0 = t(\gamma j)$ and $uj1 = s(\gamma j)$. We also have that $p_0 i : A(\gamma 0)[\varphi \mapsto u0i]$, and since $p_0 0 = t(\gamma 0)$ and $p_0 1 = s(\gamma 0)$, we can say that

$$p_0 i : A(\gamma 0)[\varphi \mapsto u0i, (1-i) \mapsto t(\gamma 0), i \mapsto s(\gamma 0)]$$

so we can use the fibrancy of A to define the term

$$q(i) \triangleq \mathbf{c}_A \gamma\ (\varphi \vee (1-i) \vee i)\ r\ (p_0 i) : \Pi(i : \mathbb{I}).A(\gamma 1)[\varphi \mapsto u1i, (1-i) \mapsto t(\gamma 1), i \mapsto s(\gamma 1)],$$

which is what we wanted. ∎

2.2.6 Interpreting glue types

We interpret Glue from CTT using Glue from $\mathcal{L}$ along with a composition operator, which we have by the following lemma:

Lemma 2.12. *Glueing is fibrant, i.e., if we have*

$$\begin{aligned} \Gamma &\vdash A \\ \Gamma &\vdash \varphi : \mathbb{I} \\ \Gamma, [\varphi] &\vdash T \\ \Gamma &\vdash w : [\varphi] \to T \to A \\ \Gamma &\vdash p : \text{isEquiv}\, w \end{aligned}$$

then there is a term $\mathbf{c} : \Phi(\Gamma; \text{Glue}\ [\varphi \mapsto (T, w)]\ A)$.

The construction of $\mathbf{c}$ in the proof of the above lemma is analogous to the construction of the composition operation for glueing in CTT [3], but formulated in $\mathcal{L}$. A crucial part of the construction is the face $\delta \triangleq \forall(\varphi \circ \gamma)$, where $\gamma : \mathbb{I} \to \Gamma$, which satisfies that $[\delta]$ implies $[\varphi(\gamma\, i)]$ for all $i : \mathbb{I}$.

2.2.7 Interpreting the universe

The universe of CTT is interpreted using the universe of fibrant types $\mathcal{U}_f$. To define the composition for the universe we follow the construction of Cohen et al. [3] in the language $\mathcal{L}$.

2.3 A concrete model of $\mathcal{L}$

Given a countable set of names let Fin be the full subcategory of Set on finite subsets of names. Let $\mathcal{C}$ be the *opposite* of the Kleisli category of the free De Morgan algebra monad on Fin. The category of *cubical sets* is the presheaf category $\widehat{\mathcal{C}}$.

It is well-known how to model dependent predicate logic in any presheaf topos, so we omit the verification of this part. We do however note how the judgements are interpreted since this will be used later on in calculations.

- A context $\Gamma \vdash$ is interpreted as a presheaf.
- The judgement $\Gamma \vdash A$ gives a pair of a presheaf Γ on $\mathcal{C}$ and a presheaf A on the category of elements of Γ.
- The judgement $\Gamma \vdash t : A$ in addition gives a global element of the presheaf A. Thus for each $I \in \mathcal{C}$ and $\gamma \in \Gamma(I)$ we have $t_{I,\gamma} \in A(I, \gamma)$.

Moreover, there is a canonical bijective correspondence between presheaves Γ on $\mathcal{C}$ and interpretations of types $\cdot \vdash \Gamma$. This justifies treating contexts as types in $\mathcal{L}$ when it is convenient to do so.

2.3.1 Assumption 1 is satisfied

Take $\mathbb{I}$ to be the functor mapping $I \mapsto \mathbf{Hom}_{\mathcal{C}}(I, 1)$, where 1 is the (globally) chosen singleton set. Since the theory of De Morgan algebras is geometric and for each I we have a De Morgan algebra together with the fact that the morphisms are De Morgan algebra morphisms, we have that $\mathbb{I}$ is an internal De Morgan algebra, as needed.

Moreover the disjunction property axiom is also geometric, and since it is clearly satisfied by each free De Morgan algebra $\mathbf{DM}(I)$, it also holds internally.

Finally, it is easy to check that we have $0 = 1 \Rightarrow \bot$ using Kripke-Joyal semantics.

2.3.2 Assumption 2 is satisfied

We will define glueing almost internally, apart from a "strictness" fix, for which we use the following lemma, which we will also use later on in Section 2.5

A strictification lemma

Lemma 2.13. *Let C be a small category and $\top$ a global element*[2] *of an object $\mathbb{K}$ in $\widehat{C}$. Denote by $[\varphi]$ the identity type $\varphi = \top$.*

Let $\Gamma \vdash \varphi : \mathbb{K}$. Suppose $\Gamma \vdash T$, $\Gamma, [\varphi] \vdash A$ and $\Gamma, [\varphi] \vdash T \cong A$ as witnessed by the terms α, β satisfying

$$\Gamma, [\varphi], x : A \vdash \alpha : T$$
$$\Gamma, [\varphi], x : T \vdash \beta : A$$

plus the equations stating that they are inverses.

Then there exists a type $\Gamma \vdash \mathcal{T}(A, T, \varphi)$ such that

1. *$\Gamma, [\varphi] \vdash \mathcal{T}(A, T, \varphi) = A$*
2. *$\Gamma \vdash T \cong \mathcal{T}(A, T, \varphi)$ by an isomorphism α', β' extending α and β. This means that the following two judgements hold.*

$$\Gamma, [\varphi], x : A \vdash \alpha = \alpha' : T$$
$$\Gamma, [\varphi], x : T \vdash \beta = \beta' : A.$$

The judgements are well-formed because in context $\Gamma, [\varphi]$ the types $\mathcal{T}(A, T, \varphi)$ and A are equal by the first item of this lemma.

3. *Let $\rho : \Delta \to \Gamma$ be a context morphism. Consider its extension $\Delta, [\varphi\rho] \to \Gamma, [\varphi]$. Then $\mathcal{T}(A, T, \varphi)\rho = \mathcal{T}(A\rho, T\rho, \varphi\rho)$.*

Proof. We write T' for $\mathcal{T}(A, T, \varphi)$ and define it as follows.

$$T'(c, \gamma) = \begin{cases} A(c, (\gamma, \star)) & \text{if } \varphi_{c,\gamma} = \top_c \\ T(c, \gamma) & \text{otherwise} \end{cases}$$

Here $\star$ is the unique proof of $[\varphi]$. The restrictions are important. Given $f : (c, \Gamma(f)(\gamma)) \to (d, \gamma)$ define $T'(f)$ by cases

$$T'(f)(x) = \begin{cases} A(f)(x) & \text{if } \varphi_d(\gamma) = \top_d(\star) \\ \beta_{c,\Gamma(f)(\gamma),\star,T(f)(x)} & \text{if } \varphi_{c,\Gamma(f)(\gamma)} = \top_c \\ T(f)(x) & \text{otherwise} \end{cases}$$

We need to check that this definition is functorial. The fact that $T'(id) = id$ is trivial. Given $f : (d, \Gamma(f)(\gamma)) \to (c, \gamma)$ and $g : (e, \Gamma(f \circ g)(\gamma)) \to (d, \Gamma(f)(\gamma))$ we have

$$T'(f \circ g)(x) = \begin{cases} A(f \circ g)(x) & \text{if } \varphi_{c,\gamma} = \top_c \\ \beta_{e,\Gamma(f\circ g)(\gamma),\star,T(f\circ g)(x)} & \text{if } \varphi_{e,\Gamma(f\circ g)(\gamma)} = \top_e \\ T(f \circ g)(x) & \text{otherwise} \end{cases}$$

In the first and third cases this is easily seen to be the same as $T'(g)(T'(f)(x))$, since if $\varphi_{e,\Gamma(f\circ g)(\gamma)} \neq \top_e$ then also $\varphi_{d,\Gamma(f)(\gamma)} \neq \top_d$ by naturality of φ and the fact that $\top$ is a global element and the terminal object is a constant presheaf.

So assume the remaining option is the case, that is, $\varphi_{e,\Gamma(f\circ g)(\gamma)} = \top_e$ but $\varphi_{c,\gamma} \neq \top_c$.

We split into two further cases.

[2]For a constructive meta-theory we add that, for each c, equality with $\top_c$ is decidable.

- Case $\varphi_{d,\Gamma(f)(\gamma)} = \top_d$. Then $T'(f)(x) = \beta_{d,\Gamma(f)(\gamma),\star,T(f)(x)}$ and so

$$T'(g)(T'(f)(x)) = T'(g)\left(\beta_{d,(\Gamma(f)(\gamma),\star,T(f)(x))}\right)$$

  By naturality of β the right-hand side is the same as

$$\beta_{e,\Gamma(f\circ g)(\gamma),\star,T(f\circ g)(x)}$$

  which is what is needed.

- Case $\varphi_{d,\Gamma(f)(\gamma)} \neq \top_d$. In this case we have

$$T'(f)(x) = T(f)(x)$$

  and

$$T'(g)(T'(f)(x)) = \beta_{e,\Gamma(f\circ g)(\gamma),\star,T(g)(T(f)(x))}$$

  which is again, as needed by functoriality of T.

Now, directly from the definition we have the equality $\Gamma, [\varphi] \vdash T' = A$.
It is similarly easy to check the last required property, the naturality of the construction.

$$T(A, T, \varphi)\rho = T(A\rho, T\rho, \varphi\rho).$$

Finally, we extend the isomorphisms α and β to α' and β'.
Define β' satisfying $\Gamma, x : T \vdash \beta' : T'$ as

$$\beta'_{c,\gamma,x} = \begin{cases} \beta_{c,\gamma,\star,x} & \text{if } \varphi_c(\gamma) = \top_c(\star) \\ x & \text{otherwise} \end{cases}$$

And α' analogously. One needs to check that this is a natural transformation, i.e., a global element. Finally, β' is the inverse to α' by construction. □

Given the following types and terms

$$\begin{aligned} \Gamma &\vdash \varphi : \mathbb{F} \\ \Gamma, [\varphi] &\vdash T \\ \Gamma &\vdash A \\ \Gamma, [\varphi] &\vdash w : T \to A \end{aligned}$$

we define a new type $\Gamma \vdash \mathrm{Glue}\ [\varphi \mapsto (T, w)]\ A$ in two steps.
First we define the type $Glue'_\Gamma(\varphi, T, A, w)$ in context Γ as

$$Glue'_\Gamma(\varphi, T, A, w) = \sum_{a:A} \sum_{t:T^\varphi} \prod_{p:[\varphi]} w(tp) = a$$

For this type we have the following property (we write G' for $Glue'(\cdots)$)

$$\Gamma, [\varphi] \vdash T \cong G'$$

with the isomorphism consisting of the second projection from right to left and from left to right we use w to construct the pair.
Finally, we define $\mathrm{Glue}\ [\varphi \mapsto (T, w)]\ A$ using Lemma 2.13 applied to the type $Glue'$. Let

$$\beta : \mathrm{Glue}\ [\varphi \mapsto (T, w)]\ A \to Glue'(\varphi, T, A, w)$$

be the extension of pairing and

$$\alpha : Glue'(\varphi, T, A, w) \to \text{Glue } [\varphi \mapsto (T, w)] \ A$$

the extension of the projection as per Lemma 2.13.

Define unglue : Glue $[\varphi \mapsto (T, w)] \ A \to A$ be the composition of β and the *first* projection $G' \to A$. Now if $\varphi = \top$ then β is just pairing and in this case we also have Glue $[\varphi \mapsto (T, w)] \ A = T$. So by definition of G' we have unglue$(t) = wt$, validating one of the equalities.

Given $\Gamma, [\varphi] \vdash t : T$ and $\Gamma \vdash a : A$ satisfying $a = wt$ on $[\varphi]$ define $\Gamma \vdash$ glue $[\varphi \mapsto t] \ a :$ Glue $[\varphi \mapsto (T, w)] \ A$ to be pairing followed by α. If $\varphi = \top$ we have, because α is just the projection in this case, that glue $[1 \mapsto t] \ a = t$.

2.3.3 Assumption 3 is satisfied

This part is subsumed by the construction in Section 2.4.3.

2.3.4 Assumption 4 is satisfied

Theorem 2.14. *$\widehat{\mathcal{C}}$ models an operation $\forall : \mathbb{F}^{\mathbb{I}} \to \mathbb{F}$ which is right-adjoint to the constant map of posets $\mathbb{F} \to \mathbb{F}^{\mathbb{I}}$.*

Proof. We will first give a concrete description of $\mathbb{I}$ and $\mathbb{F}$. We know that $\mathbb{I}(n) = DM(n)$. We use Birkhoff duality [2] between finite distributive lattices and finite posets. This duality is given by a functor $J = \mathbf{Hom}_{\text{fDL}}(-, 2)$ from finite distributive lattices to the opposite of the category of finite posets. This functor sends a distributive lattice to its join-irreducible elements. It's inverse is the functor $\mathbf{Hom}_{\text{poset}}(-, 2)$ which sends a poset to its the distributive lattice of lower sets. This restricts to a duality between free distributive lattices and powers of 2. Every free *De Morgan* algebra on n generators is a free distributive lattice on $2n$ generators. We obtain a duality with the category of *even* powers of 2 and maps preserving the De Morgan involution [4]. Moreover, this duality is poset enriched: If $\psi \leq \varphi : DM(n) \to DM(m)$, then the corresponding maps on even powers of 2, which are defined by pre-composition, are in the same order relation.

The dual of the inclusion map is the projection $p : 2^{2(n+1)} \to 2^{2n}$. This has a right adjoint: concatenation with 11: $p\alpha \leq \beta$ iff $\alpha \leq \beta \cdot 11$. Concatenation with 11 is natural:

$$\begin{array}{ccc} 2^{2n} & \xrightarrow{f} & 2^{2m} \\ \uparrow\downarrow 11 & & \uparrow\downarrow 11 \\ 2^{2(n+1)} & \xrightarrow{(f,id)} & 2^{2(m+1)} \end{array}$$

By duality we obtain a natural right adjoint to the poset-inclusion of DM-algebras. Finally, we recall that in $\widehat{\mathcal{C}}$ we have $\mathbb{I}^{\mathbb{I}}(n) = \mathbb{I}(n+1)$ and hence we have an internal map $\forall : \mathbb{I}^{\mathbb{I}} \to \mathbb{I}$ which is right-adjoint to the constant map $\mathbb{I} \to \mathbb{I}^{\mathbb{I}}$.

In [3] the face lattice is defined as the quotient of $\mathbb{I}$ by the congruence $x \wedge 1 - x = 0$, for all x. In cubical sets these two definitions coincide. Let us temporarily write $\mathbb{F}_q$ for the image of $\mathbb{I}$ in Ω. Since $\mathbb{F}$ satisfies $[i] \wedge [1 - i] = \bot$, because $0 \neq 1$. So, we have a surjective lattice map: $\mathbb{F}_q \to \mathbb{F}$. We will show that it is also injective. Let φ, ψ be (generalised) elements of $\mathbb{I}$. Suppose that $[\psi] = [\varphi]$, i.e. $\psi = 1$ iff $\varphi = 1$. Then the sieves of functions in $\mathcal{C}$ which evaluate ψ, φ to 1 are the same. So, we may assume that ψ, φ have the same free variables. By considering the disjunctive normal forms of φ, ψ we see they should have the same conjuncts upto the equality $x \wedge -x = 0$. Injectivity follows. Hence, $\mathbb{F}_q = \mathbb{F}$.

The quotient presentation has the advantage of having decidable equality (externally) in the model, and hence this is used in the implementation. It is also geometric since the construction of a distributive lattice by generators (from $\mathbb{I}$) and relations can be done by considering a quotient of $\mathcal{F}\mathcal{F}(\mathbb{I})$, where $\mathcal{F}$ denote the (Kuratowski) finite power set. Both quotients and $\mathcal{F}$ which are geometric. This will be useful later.

As just explained $\mathbb{F}$ is the quotient of $\mathbb{I}$ by the relation $x \wedge 1 - x = 0$, for all x. This is a geometric formula and hence holds at each stage n. As duality turns the quotients into inclusions, we have the inclusion $\{01, 10, 11\}^n \subset \mathbb{2}^{2n}$ as the set of join irreducible elements; as 00 presents $x \wedge -x$ which is now identified with $\bot$ and hence no longer join-irreducible. This presentation allows us to define $\forall : \mathbb{F}^{\mathbb{I}} \to \mathbb{F}$. Since $\mathbb{F}^{\mathbb{I}}(n) = \mathbb{F}(n+1)$, so the right adjoint is again given by concatenation by 11. We just replace $\mathbb{2}^2$ by $\{01, 10, 11\}$ in the diagram above. □

2.4 More models of $\mathcal{L}$

Lemma 2.15. *Let $\mathbb{C}, \mathbb{D}$ be small categories and let $\pi : \mathbb{D} \times \mathbb{C} \to \mathbb{D}$ be the projection functor. Then the geometric morphism $\pi^* \dashv \pi_*$ is open. If $\mathbb{C}$ is inhabited then it is also surjective.*

Proof. By Theorem C.3.1.7 of [6] it suffices to show that π^* is sub-logical. We use Lemma C.3.1.2 of loc. cit. to show this (we use the notation introduced in that lemma).

Let $b : \pi(I, n) \to J$ be a morphism in $\mathbb{D}$. Let $U' = (J, n)$, $a = (b, id_n) : (I, n) \to (J, n)$, $r = id_J : \pi U' \to J$ and $i = id_J : J \to \pi U'$. Then we have $r \circ i = id_J$ and $i \circ b = \pi a$ as required by Lemma C.3.1.2.

If $\mathbb{C}$ is inhabited the projection π is surjective on objects, so the corresponding geometric morphism is surjective; see [6, A4.2.7b] □

The previous lemma may be read as $\widehat{\mathbb{D} \times \mathbb{C}}$ is a conservative extension of $\hat{\mathbb{D}}$, provided $\mathbb{C}$ is inhabited.

Lemma 2.16. *If $\mathbb{C}$ has an initial object 0, then π^* is full, faithful, and cartesian closed.*

Proof. The functor π has a *left* adjoint, which is the functor

$$\begin{aligned} \iota : \mathbb{D} &\to \mathbb{D} \times \mathbb{C} \\ \iota(I) &= (I, 0) \end{aligned}$$

Trivially we have $\pi \circ \iota = id_{\mathbb{D}}$. Thus we have that ι^* is *left* adjoint to π^* and because $\pi \circ \iota = id_{\mathbb{D}}$ we also have $\iota^* \circ \pi^* = id$ and moreover the counit of the adjunction is the identity. Hence the functor π^* is full and faithful [7, Theorem IV.3.1] and by [6, Corollary A.1.5.9], since ι^* preserves all limits, we have that π^* cartesian closed. □

Let $\Omega^{\mathbb{C}}$ be the subobject classifier of $\widehat{\mathcal{C} \times \mathbb{C}}$ and $\Omega^{\mathcal{C}}$ the subobject classifier of $\widehat{\mathcal{C}}$.

Lemma 2.17. *There is a* monomorphism $\upsilon : \pi^* \left(\Omega^{\mathcal{C}}\right) \to \Omega^{\mathbb{C}}$ *which fits into the pullback*

$$\begin{array}{ccc} \pi^*(1) & \xrightarrow{\cong} & 1 \\ {\scriptstyle \pi^*(\mathrm{true})}\downarrow & \lrcorner & \downarrow{\scriptstyle \mathrm{true}} \\ \pi^*\left(\Omega^{\mathcal{C}}\right) & \xrightarrow{\upsilon} & \Omega^{\mathbb{C}} \end{array}$$

Proof. Since π^* preserves monos $\pi^*(\mathrm{true})$ is a mono, hence define υ to be its characteristic map. Concretely it maps

$$\upsilon_{I,c}(S) = \{(f, g) \mid f \in S\}$$

so it is clearly a mono. □

Corollary 2.18. *If $X = \pi^*(Y)$ then the equality predicate $\chi_\delta : X \times X \to \Omega^{\mathbb{C}}$ factors uniquely through υ and the inclusion of the equality predicate of Y.*

Proof. The equality predicate is by definition the characteristic map of the diagonal $\delta : X \to X \times X$. Let $\delta' : Y \to Y \times Y$ be the diagonal. Because π^* preserves finite limits the following square is a pullback.

$$\begin{array}{ccccc} X & \longrightarrow & \pi^*(1) & \xrightarrow{\cong} & 1 \\ {\scriptstyle \delta=\pi^*(\delta')}\downarrow & \lrcorner & \downarrow{\scriptstyle \pi^*(\mathsf{true})} & \lrcorner & \downarrow{\scriptstyle \mathsf{true}} \\ X \times X & \xrightarrow{\pi^*(\chi_{\delta'})} & \pi^*\left(\Omega^{\mathcal{C}}\right) & \xrightarrow{\upsilon} & \Omega^{\mathbb{C}} \end{array}$$

and by uniqueness of characteristic maps we have $\upsilon \circ \pi^*\left(\chi_{\delta'}\right) = \chi_\delta$. Uniqueness of the factorisation follows from the fact that υ is a mono. □

Let $\mathbb{C}$ be a category with an initial object. We now show that $\widehat{\mathcal{C} \times \mathbb{C}}$ models $\mathcal{L}$.

2.4.1 Assumption 1 is satisfied

Let $\mathbb{I}^{\mathbb{C}} = \pi^*(\mathbb{I})$. Since π^* preserves products we can lift all the De Morgan algebra operations of $\mathbb{I}$ to operations on $\mathbb{I}^{\mathbb{C}}$. The theory of a De Morgan algebra with a disjunction property and $0 \neq 1$ is geometric [8, Section X.3]. Thus the geometric morphism $\pi^* \dashv \pi_*$ preserves validity of all the axioms, which means that $\mathbb{I}^{\mathbb{C}}$ is an internal De Morgan algebra with $0 \neq 1$ and the disjunction property.

Faces

Lemma 2.19. *Let $\mathbb{F}^{\mathbb{C}} \in \widehat{\mathcal{C} \times \mathbb{C}}$ and $\mathbb{F} \in \widehat{\mathcal{C}}$ be defined as in Section 2.1.2 from $\mathbb{I}^{\mathbb{C}}$ and $\mathbb{I}$. Then $\mathbb{F}^{\mathbb{C}} \cong \pi^*(\mathbb{F})$.*

Proof. Let $e : \mathbb{I}^{\mathbb{C}} \to \Omega^{\mathbb{C}}$ be the composition $\chi_\delta \circ \langle id, 1\rangle$ where δ is the diagonal $\mathbb{I}^{\mathbb{C}} \to \mathbb{I}^{\mathbb{C}} \times \mathbb{I}^{\mathbb{C}}$. By definition $\mathbb{F}^{\mathbb{C}}$ is the image of e. By Corollary 2.18 and the way we have defined $\mathbb{I}^{\mathbb{C}}$ and all the operations on it we have that $e = \upsilon \circ \pi^*(e')$ where $e' : \mathbb{I} \to \Omega^{\mathcal{C}}$ is defined analogously to e above.

By definition $\mathbb{F}$ is the image of e'. Because inverse images of geometric morphisms preserve image factorisations $\pi^*(\mathbb{F})$ is the image of $\pi^*(e')$. Finally, because υ is a mono the image of $\upsilon \circ \pi^*(e')$ is canonically isomorphic to the image of $\pi^*(e')$, which is what the lemma claims. □

2.4.2 Assumption 2 is satisfied

This proceeds exactly as in Section 2.3.

2.4.3 Assumption 3 is satisfied

Lemma 2.20. *Let $\mathbb{C}$ and $\mathbb{D}$ be small categories and assume $\mathbb{D}$ has products. Let $k_1 : \mathbb{D} \to \widehat{\mathbb{D}}$ and $k_2 : \mathbb{C} \times \mathbb{D} \to \widehat{\mathbb{C} \times \mathbb{D}}$ be the Yoneda embeddings. Let $\pi^* : \widehat{\mathbb{D}} \to \widehat{\mathbb{C} \times \mathbb{D}}$ be the constant presheaf functor.*

For any $d, e \in \mathbb{D}$ and any $c \in \mathbb{C}$ there is an isomorphism

$$k_2(c, d) \times \pi^*(k_1 e) \cong k_2(c, d \times e)$$

in $\widehat{\mathbb{C} \times \mathbb{D}}$ which is natural in c, d and e.

Proof. For any $(c', d') \in \mathbb{C} \times \mathbb{D}$

$$\begin{aligned} (k_2(c, d) \times \pi^*(k_1 e))(c', d') &= \mathbf{Hom}_{\mathbb{C}\times\mathbb{D}}\left((c', d'), (c, d)\right) \times \mathbf{Hom}_{\mathbb{D}}\left(d', e\right) \\ &\cong \mathbf{Hom}_{\mathbb{D}}\left(d', d\right) \times \mathbf{Hom}_{\mathbb{C}}\left(c', c\right) \times \mathbf{Hom}_{\mathbb{D}}\left(d', e\right) \end{aligned}$$

and because the hom functor preserves products we have

$$\begin{aligned} &\cong \mathbf{Hom}_{\mathbb{D}}\left(d', d \times e\right) \times \mathbf{Hom}_{\mathbb{C}}\left(c', c\right) \\ &\cong \mathbf{Hom}_{\mathbb{C}\times\mathbb{D}}\left((c', d'), (c, d \times e)\right) \\ &= k_2(c, d \times e)(c', d') \end{aligned}$$

as required. □

The following lemma is useful for describing the composition externally, which is needed do define the fibrant universe.

Lemma 2.21. *Let $\mathbb{C}$ be a small category. Let $\mathbb{I}^{\mathbb{C}} \in \widehat{\mathbb{C} \times \mathcal{C}}$ be the inclusion $\pi^*(\mathbb{I})$ of $\mathbb{I} \in \widehat{\mathcal{C}}$. Let $X \in \widehat{\mathcal{C} \times \mathbb{C}}$. Then for any $c \in \mathbb{C}$, any $I \in \mathcal{C}$ and any $i \notin I$ we have*

$$X^{\mathbb{I}^{\mathbb{C}}}(I, c) \cong X((I, i), c)$$

naturally in c, I and i.

Remark 2.22. Note that disjoint union is the *coproduct* in the Kleisli category of the free De Morgan algebra monad. Hence disjoint union is the product in $\mathcal{C}$.

Proof. Using the Yoneda lemma and the defining property of exponents we have

$$\begin{aligned} X^{\mathbb{I}^{\mathbb{C}}}(I, c) &\cong \mathbf{Hom}_{\widehat{\mathcal{C}\times\mathbb{C}}}\left(y(I, c), X^{\mathbb{I}^{\mathbb{C}}}\right) \\ &\cong \mathbf{Hom}_{\widehat{\mathcal{C}\times\mathbb{C}}}\left(y(I, c) \times \pi^*(\mathbb{I}), X\right) \end{aligned}$$

which by Lemma 2.20, together with the fact that $\mathbb{I}$ is isomorphic to $y\{i\}$, is isomorphic to

$$\begin{aligned} &\cong \mathbf{Hom}_{\widehat{\mathcal{C}\times\mathbb{C}}}\left(y(I \cup \{i\}, c), X\right) \\ &\cong X(I \cup \{i\}, c). \end{aligned}$$

Concretely, the isomorphism $\alpha^c_{I,i}$ maps $\xi \in X^{\mathbb{I}^{\mathbb{C}}}(I, c)$ to $\xi_{(\iota_{I,i}, id_c,)}(i)$, where $\iota_{I,i} : I \to I, i$ (in $\mathcal{C}^{\mathrm{op}}$) is the inclusion. Its inverse $\beta^c_{I,i}$ maps $x \in X(I \cup \{i\}, c)$ to the family of functions $\xi_{(f,g)} : \mathbb{I}(J) \to X(J, d)$ indexed by morphisms $(f, g) : (J, d) \to (I, c)$ (in $(\mathcal{C} \times \mathbb{C})$). This family is defined as

$$\xi_{(f,g)}(\varphi) = X([g, i \mapsto \varphi], g)(x)$$

where $[g, i \mapsto \varphi]$ is the map $I, i \to J$ (in $\mathcal{C}^{\mathrm{op}}$) which maps i to φ and otherwise acts as g. This map is well-defined because disjoint union is the coproduct in $\mathcal{C}^{\mathrm{op}}$. □

We can now define the universe $\mathcal{U}^{\mathbb{C}}_f$. First, the Hofmann-Streicher universe $\mathcal{U}^{\mathbb{C}}$ in $\widehat{\mathcal{C} \times \mathbb{C}}$ maps (I, c) to the set of functors valued in $\mathfrak{U}$ on the category of elements of $y(I, c)$. It acts on morphisms $(I, c) \to (J, d)$ by composition (in the same way as substitution in types is modelled).

The elements map

$$\frac{\Gamma \vdash a : \mathcal{U}^{\mathbb{C}}}{\Gamma \vdash \mathsf{El}(a)}$$

is interpreted as

$$\mathsf{El}(a)((I, c), \gamma) = a_{(I,c),\gamma}(\star)\,(id_{I,c})\,,$$

recalling that terms are interpreted as global elements, and $\star$ is the unique inhabitant of the chosen singleton set.

We define $\mathcal{U}^{\mathbb{C}}_f$ analogously to the way it is defined in Section 2.3, that is

$$\mathcal{U}^{\mathbb{C}}_f(I, c) = \mathrm{Ty}(y(I, c)).$$

We first look at the following rule.

$$\frac{\Gamma \vdash a : \mathcal{U}^{\mathbb{C}} \qquad \vdash \mathbf{c} : \Phi(\Gamma; \mathrm{El}(a))}{\Gamma \vdash (\!| a, \mathbf{c} |\!) : \mathcal{U}^{\mathbb{C}}_f}$$

Let us write $b = (\!|a, \mathbf{c}|\!)$. We need to give for each $I \in \mathcal{C}$, $c \in \mathbb{C}$ and $\gamma \in \Gamma(I, c)$ a pair (b_0, b_1) where

$$y(I,c) \vdash b_0 : \mathcal{U}^{\mathbb{C}}$$
$$\cdot \vdash b_1 : \Phi(y(I,c); \mathrm{El}(b_0))$$

Now b_0 is easy. It is simply $a_{(I,c),\gamma}$. Composition is also conceptually simple, but somewhat difficult to write down precisely. Elements $\gamma \in \Gamma(I, c)$ are in bijective correspondence (by Yoneda and exponential transpose) to terms $\overline{\gamma}$

$$\cdot \vdash \overline{\gamma} : y(I,c) \to \Gamma.$$

Thus we define

$$b_1 = \lambda \rho . \mathbf{c} \left(\overline{\gamma} \circ \rho \right).$$

One checks that this is well-defined and natural by a tedious computation, which we omit here.

We now look at the converse rule in $\mathcal{L}$

$$\frac{\Gamma \vdash a : \mathcal{U}_f}{\Gamma \vdash \mathrm{El}(a)} \qquad\qquad \frac{\Gamma \vdash a : \mathcal{U}_f}{\vdash \mathrm{Comp}(a) : \Phi(\Gamma; \mathrm{El}(a))}.$$

To interpret this rule with $\mathcal{U}_f^{\mathbb{C}}$, we interpret for any a and $\mathbf{c}$, $\mathrm{El}((\!|a, \mathbf{c}|\!))$ by $\mathrm{El}(a)$, where the latter is El map of the Hofmann-Streicher universe.

We need to define $\mathrm{Comp}(a)$ which we abbreviate to c. We need to give for each $I \in \mathcal{C}$ and $c \in \mathbb{C}$ an element $c_{I,c} \in \Phi(\Gamma; \mathrm{El}(a))(I, c)$, and this family needs to be natural in I and c. Given $\gamma \in (\Gamma^{\mathbb{I}^{\mathbb{C}}})(I, c)$ and a fresh $i \notin I$ we get by Lemma 2.21 an element $\gamma' \in \Gamma\left((I, i), c\right)$. Let $\overline{\gamma'} : y((I, i), c) \to \Gamma$ be the morphism corresponding to γ' by the Yoneda lemma. Thus we get from a the term $c'_{I,i,c,\gamma}$

$$\cdot \vdash c'_{I,i,c,\gamma} : \Phi(y((I,i),c); \mathrm{El}(a)\overline{\gamma'})$$

and hence by weakening a term

$$y(I,c) \vdash c'_{I,i,c,\gamma} : \Phi(y((I,i),c); \mathrm{El}(a)\overline{\gamma'})$$

By Lemma 2.20 and the way $\mathbb{I}^{\mathbb{C}}$ is defined we have a canonical isomorphism $y((I, i), c) \cong y(I, c) \times \mathbb{I}^{\mathbb{C}}$. We now apply $c'_{I,i,c,\gamma}$ to the path $\delta = \lambda(i : \mathbb{I}^{\mathbb{C}}).(\rho, i)$ to get the element

$$\rho : y(I,c) \vdash c'_{I,i,c,\gamma}\delta : \Pi(\varphi : \mathbb{F})(u : \Pi(i : \mathbb{I}).\, [\varphi] \to B(\delta(i))).B(\delta(0))[\varphi \mapsto u(0)] \to B(\delta(1))[\varphi \mapsto u(1)]$$

Where $B = \mathrm{El}(a)\overline{\gamma'}$.

From this element we can define $c_{I,c}$ by using the Yoneda lemma again to get the element $\overline{c'_{I,i,c,\gamma}}$ of type

$$\Pi(\varphi : \mathbb{F})(u : \Pi(i : \mathbb{I}).\, [\varphi] \to B(\delta(i))).B(\delta(0))[\varphi \mapsto u(0)] \to B(\delta(1))[\varphi \mapsto u(1)],$$

which is a type in context $y(I, c)$, at $(I, c), id_{I,c}$. To recap, the composition c will map $\gamma \in (\Gamma^{\mathbb{I}^{\mathbb{C}}})(I, c)$ to the element $\overline{c'_{I,i,c,\gamma}}$.

Lemma 2.23. *For any a and $\mathbf{c}$ of correct types we have*

$$\mathrm{Comp}((\!|a, \mathbf{c}|\!)) = \mathbf{c}$$
$$\mathrm{El}((\!|a, \mathbf{c}|\!)) = \mathrm{El}(a)$$
$$(\!|\mathrm{El}(a), \mathrm{Comp}(a)|\!) = a$$

2.4.4 Assumption 4 is satisfied

Using Lemmas 2.16 and 2.19 we can define $\forall$ in $\widehat{\mathcal{C}\times\omega}$ as the inclusion of the $\forall$ from $\widehat{\mathcal{C}}$. Lemma 2.15 can then be used to show that the new $\forall$ is the right adjoint to the map $\varphi \mapsto \lambda\_.\varphi$.

2.5 A model of GCTT

The construction of the model of GCTT uses the internal language, in the form of *dependent predicate logic* with additional types, terms, and equalities corresponding to objects, arrows and properties of the particular category, of the presheaf topos $\widehat{\mathcal{C}\times\omega}$. Thus the internal language we use is an extension of the language $\mathcal{L}$ used above.

We define our model of GCTT as an extension of the model of CTT from section 2.2. Therefore we only need to show how to interpret the new rules of GCTT, i.e., the ones that have to do with *guarded recursive types*.

2.5.1 The functor $\triangleright$

We first define $\triangleright$ on $\widehat{\mathbb{C}\times\omega}$ and then extend it to types in context and delayed substitutions.

Given $X \in \widehat{\mathbb{C}\times\omega}$ we define

$$\triangleright X(I,n) = \begin{cases} 1 & \text{if } n=0 \\ X(I,m) & \text{if } n=m+1 \end{cases}$$

with restrictions inherited from X as in if $(f, n\le m) : (I,n) \to (J,m)$ then

$$\triangleright X(f, n\le m) : X(J,m) \to X(I,n)$$
$$\triangleright X(f, n\le m) = \begin{cases} ! & \text{if } n=1 \\ X(f, k\le m-1) & \text{if } n=k+1 \end{cases}$$

where $n\le m$ is the unique morphism $n\to m$ (and similarly $k\le m-1$), and ! as the unique morphism into 1, the chosen singleton set.

There is a natural transformation

$$\text{next} : id_{\widehat{\mathbb{C}\times\omega}} \to \triangleright$$
$$(\text{next}_X)_{I,0} = !$$
$$(\text{next}_X)_{I,n+1} = X\left(id_I, (n\le n+1)\right).$$

Lemma 2.24. *The functor $\triangleright$ is continuous.*

Proof. Limits in presheaf toposes are computed pointwise. Limit of any diagram of terminal objects is the terminal object. ☐

Lemma 2.25. *For any X and any morphism $\alpha : \triangleright X \to X$ there exists a unique global element $\beta : 1 \to X$ such that*

$$\alpha \circ \text{next} \circ \beta = \beta.$$

Hence the triple $(\widehat{\mathbb{C}\times\omega}, \triangleright, \text{next})$ is a model of guarded recursive terms [1, Definition 6.1].

Proof. Any global element β satisfying the fixed-point equation must satisfy the following two equations

$$\beta_{I,0}(\star) = \alpha_{I,0}(\star)$$
$$\beta_{I,n+1}(\star) = \alpha_{I,n+1}\left(\beta_{I,n}(\star)\right).$$

Hence define β recursively on n. It is then easy to see by induction on n that β is a global element and that it satisfies the fixed-point equation. ☐

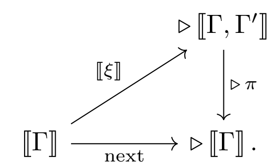
Using [1, Theorem 6.3] $\triangleright$ extends to all slices of $\widehat{\mathbb{C} \times \omega}$ and contractive morphisms on slices have unique fixed-points.

Remark 2.26. The construction in [1] ignores coherence issues, and there are no delayed substitutions, so we will define $\triangleright$ for types in contexts again, but the theorem cited gives us assurance that $\triangleright$ and fixed points exist in all slices. Moreover inspection of the construction in the cited paper shows that we are defining the correct notion, up to equivalent presentation of slices.

Delayed substitutions Semantically a *delayed* substitution

$$\vdash \xi : \Gamma \twoheadrightarrow \Gamma'$$

will be interpreted as a morphism $[\![\xi]\!] : [\![\Gamma]\!] \to \triangleright [\![\Gamma, \Gamma']\!]$ making the following diagram commute



Here $\pi : [\![\Gamma, \Gamma']\!] \to [\![\Gamma]\!]$ is the composition of projections of the form $[\![\Gamma, \Gamma'', x : A]\!] \to [\![\Gamma, \Gamma'']\!]$.

In particular, if Γ' is the empty context then $\pi = id_{[\![\Gamma]\!]}$ and so $[\![\cdot]\!] = \text{next}$, where $\cdot$ is the empty delayed substitution.

Thus given a delayed substitution $\vdash \xi : \Gamma \twoheadrightarrow \Gamma'$ and a type

$$\Gamma, \Gamma' \vdash A$$

define

$$\Gamma \vdash \triangleright \xi . A$$

to be

$$(\triangleright \xi . A)(I, n, \gamma) = \begin{cases} 1 & \text{if } n = 0 \\ A\Big(I, m, [\![\xi]\!]_{I,n}(\gamma)\Big) & \text{if } n = m + 1 \end{cases}$$

Note that this is exactly like substitution $A\xi$, except in the case where $n = 0$.

In turn, we interpret the rules

$$\frac{\Gamma \vdash}{\vdash \cdot : \Gamma \twoheadrightarrow \cdot} \qquad \frac{\vdash \xi : \Gamma \twoheadrightarrow \Gamma' \qquad \Gamma \vdash t : \triangleright \xi . A}{\vdash \xi\,[x \leftarrow t] : \Gamma \twoheadrightarrow \Gamma', x : A}$$

as follows. First, the empty delayed substitution is interpreted as next, as we already remarked above. Given $\vdash \xi : \Gamma \twoheadrightarrow \Gamma'$ and $\Gamma \vdash t : \triangleright \xi . A$ define

$$[\![\vdash \xi\,[x \leftarrow t] : \Gamma \twoheadrightarrow \Gamma', x : A]\!]_{I,n}(\gamma) = \begin{cases} \star & \text{if } n = 0 \\ (\xi_{I,n}(\gamma), t_{I,n,\gamma}(\star)) & \text{otherwise} \end{cases}$$

The following two rules are easy to verify with the definitions we have provided.

$$\frac{\vdash \xi\,[x \leftarrow t] : \Gamma \twoheadrightarrow \Gamma', x : B \qquad \Gamma, \Gamma' \vdash A}{\Gamma \vdash \triangleright \xi\,[x \leftarrow t] . A = \triangleright \xi . A}$$

$$\frac{\vdash \xi\,[x \leftarrow t, y \leftarrow u]\,\xi' : \Gamma \twoheadrightarrow \Gamma', x : B, y : C, \Gamma'' \qquad \Gamma, \Gamma' \vdash C \qquad \Gamma, \Gamma', x : B, y : C, \Gamma'' \vdash A}{\Gamma \vdash \triangleright \xi\,[x \leftarrow t, y \leftarrow u]\,\xi' . A = \triangleright \xi\,[y \leftarrow u, x \leftarrow t]\,\xi' . A}$$

The first rule follows immediately by observing that the interpretation of A in the extended context $\Gamma, \Gamma', x : B$ ignores the last component. The second rule follows because we exchange the same components in the interpretation of A as well as in the interpretation of the delayed substitution.

For the rest we need to define how `next` is interpreted.

Next

$$\frac{\Gamma, \Gamma' \vdash t : A \qquad \vdash \xi : \Gamma \rightarrow \Gamma'}{\Gamma \vdash \mathsf{next}\, \xi.t : \triangleright\xi.A}$$

Given a term t and a delayed substitution ξ we define

$$[\![\mathsf{next}\, \xi.t]\!]_{I,n,\gamma}(\star) = \begin{cases} \star & \text{if } n = 0 \\ t_{I,m,[\![\xi]\!]_{I,n}(\gamma)}(\star) & \text{if } n = m+1 \end{cases}$$

With this it is easy to see by computation that the rule

$$\frac{\vdash \xi : \Gamma \rightarrow \Gamma' \qquad \Gamma, \Gamma', x : B \vdash A \qquad \Gamma, \Gamma' \vdash t : B}{\Gamma \vdash \triangleright\xi \left[x \leftarrow \mathsf{next}\, \xi.t\right].A = \triangleright\xi.A[t/x]}$$

is sound.

$\triangleright$ **and** Π As we mentioned above a delayed substitution ξ is a morphism $\Gamma \to \triangleright(\Gamma, \Gamma')$. Hence we can treat it as a term of type $\triangleright(\Gamma, \Gamma')$ in context Γ. Further given a morphism $\gamma : \mathbb{I}^\omega \to \Gamma$ we can form the morphism

$$\xi \circ \gamma : \mathbb{I}^\omega \to \triangleright(\Gamma, \Gamma').$$

Finally by using Proposition 2.31 we can transport $\xi \circ \gamma : \mathbb{I}^\omega \to \triangleright(\Gamma, \Gamma')$ to a term

$$\overline{\xi \circ \gamma} : \triangleright(\mathbb{I}^\omega \to \Gamma, \Gamma')$$

in the empty context.

Lemma 2.27. *Given γ and ξ as above then for any type $\Gamma, \Gamma' \vdash A$ we have the equality of types*

$$i : \mathbb{I}^\omega \vdash \triangleright\left[\gamma' \leftarrow \overline{\gamma \circ \xi}\right].A\left(\gamma'(i)\right) = \triangleright\xi\gamma(i).A\left(\gamma(i)\right).$$

Here $\xi\gamma(i)$ is the delayed substitution $\vdash : \mathbb{I}^\omega \rightarrow \Gamma, \Gamma'$ *obtained by substitution in terms of ξ.*

Proof. Proof by computation. Requires the unfolding of the definition of the isomorphism in Proposition 2.31. ☐

Dependent products and "constant" types To define composition for the $\triangleright$ type we will need type isomorphism commuting $\triangleright$ and dependent products in certain cases. We start with a definition.

Definition 2.28. A type $\Gamma \vdash A$ is *constant with respect to ω* if for all $I \in \mathbb{C}, n \in \omega, \gamma \in \Gamma(I, n)$ and for all $m \leq n$ the restriction

$$A(I, n, \gamma) \to A\left(I, m, \Gamma(id_I, m \leq n)(\gamma)\right)$$

is the *identity function* (in particular, the two sets are equal).

Below we will use the shorter notation $\gamma_{\restriction_m}$ for $\Gamma(id_I, \leq n)(\gamma)$.

Note that this is a direct generalisation of "being constant" (being in the image of π^*) for presheaves (i.e., closed types). We have the following easy, but important, lemma.

Lemma 2.29. *Being constant with respect to ω is closed under substitution. If $\Gamma \vdash A$ is constant and $\rho : \Gamma' \to \Gamma$ is a context morphism then $\Gamma' \vdash A\rho$ is constant.*

Lemma 2.30. *Let X be a presheaf in the* essential image *of π^*. The identity type $x : X, y : X \vdash \mathrm{Id}_X(x, y)$ is* constant *with respect to ω.*

Proof. Recall that we have for $\gamma, \gamma' \in X(I,n)$.

$$(\mathrm{Id}_X(x,y))(I,n,\gamma,\gamma') = \begin{cases} 1 & \text{if } \gamma = \gamma' \\ \emptyset & \text{otherwise} \end{cases}$$

Thus for any $m \leq n$

$$(\mathrm{Id}_X(x,y))(I,m,\gamma_{\restriction_m},\gamma'_{\restriction_m}) = \begin{cases} 1 & \text{if } \gamma_{\restriction_m} = \gamma'_{\restriction_m} \\ \emptyset & \text{otherwise} \end{cases}$$

But since $\cdot_{\restriction_m}$ is an isomorphism we have $\gamma_{\restriction_m} = \gamma'_{\restriction_m}$ if and only if $m = m'$, which concludes the proof. Since all the sets are singletons or empty the relevant restriction is then trivially the identity function. □

Using the assumptions stated above we have the following proposition.

Proposition 2.31. *Assume*

$$\begin{aligned} &\Gamma \vdash A \\ &\Gamma, \Gamma', x : A \vdash B \\ &\vdash \xi : \Gamma \rightharpoonup \Gamma' \end{aligned}$$

and further that A is constant with respect to ω*.*

The canonical morphism from left to right in

$$\Gamma \vdash \triangleright\xi.\Pi(x:A).B \cong \Pi(x:A).\triangleright\xi.B \tag{1}$$

is an isomorphism. The canonical morphism is derived from the term $\lambda f.\lambda x.\,\mathsf{next}\,[\xi, f' \leftarrow f]\,.(f'\,x)$.

Proof. We need to establish an isomorphism of two presheaves on the category of elements of Γ. Since we already have one of the directions we will first define the other direction explicitly. We define $F : \Pi(x : A).\triangleright\xi.B \to \triangleright\xi.\Pi(x:A).B$. Let $I \in \mathbb{C}$, $n \in \omega$ and $\gamma \in \Gamma(I,n)$. Take $\alpha \in (\Pi(x:A).\triangleright\xi.B)\,(I,n,\gamma)$. If $n = 0$ then we have only one choice.

$$F_{I,0,\gamma}(\alpha) = \star$$

So assume that $n = m+1$. Then we need to provide an element of

$$F_{I,n,\gamma}(\alpha) \in (\Pi(x:A).B)\,(I,m,\xi_{I,n}(\gamma))\,.$$

Which means that for each $f : J \to I$ and each $k \leq m$ we need to give a dependent function

$$\beta_{f,k} : (a \in A\,(J,k,(\Gamma,\Gamma')(f,k\leq m)\,(\xi_{I,n}(\gamma)))) \to B\,(J,k,(\Gamma,\Gamma')(f,k\leq m)\,(\xi_{I,n}(\gamma))\,,a)$$

Because $\Gamma \vdash A$ we have

$$A\,(J,k,(\Gamma,\Gamma')(f,k\leq m)\,(\xi_{I,n}(\gamma))) = A\,(J,k,\pi_{J,k}\,((\Gamma,\Gamma')(f,k\leq m)\,(\xi_{I,n}(\gamma))))$$

where $\pi : \Gamma, \Gamma' \to \Gamma$ is the composition of projections. By naturality we have

$$\pi_{J,k}\,((\Gamma,\Gamma')(f,k\leq m)\,(\xi_{I,n}(\gamma))) = \Gamma(f,k\leq m)\,(\pi_{I,m}\,(\xi_{I,n}(\gamma)))\,.$$

Now $\pi_{I,m} = \triangleright(\pi)_{I,n}$ and so we have (because ξ is a delayed substitution)

$$\pi_{I,m}\,(\xi_{I,n}(\gamma)) = \mathrm{next}(\gamma)_{I,n} = \Gamma(id_I, m \leq n)(\gamma).$$

Hence we have

$$A\left(J,k,(\Gamma,\Gamma')(f,k\le m)\left(\xi_{I,n}(\gamma)\right)\right)=A\left(J,k,\Gamma(f,k\le n)(\gamma)\right).$$

And because A is *constant* we further have

$$A\left(J,k,\Gamma(f,k\le n)(\gamma)\right)=A(J,k+1,\Gamma(f,k+1\le n)(\gamma))$$

(by assumption $k\le m$ and $n=m+1$.

Now $\alpha_{f,k+1}$ is a dependent function

$$(a\in A(J,k+1,\Gamma(f,k+1\le n)(\gamma)))\to(\triangleright\xi.B)(J,k+1,\Gamma(f,k+1\le n)(\gamma),a)$$

And we have

$$\triangleright\xi.B)\left(J,k+1,\Gamma(f,k+1\le n)(\gamma),a\right)=B\left(J,k,\xi_{J,k+1}(\Gamma(f,k+1\le n)(\gamma)),a\right)$$

(because the relevant restriction of A is the identity). Now

$$\begin{aligned}\xi_{J,k+1}(\Gamma(f,k+1\le n))&=(\triangleright(\Gamma,\Gamma'))(f,k+1\le n)(\xi_{I,n}(\gamma))\\&=(\Gamma,\Gamma')(f,k\le m)(\xi_{I,n}(\gamma)).\end{aligned}$$

Thus, we can define

$$\beta_{f,k}=\alpha_{f,k+1}.$$

The fact that β is a natural family follows from the fact that α is a natural family. Naturality of F follows easily by the fact that restrictions for Π types are defined by "precomposition".

The fact that it is the inverse to the canonical morphism follows by a tedious computation. □

Corollary 2.32. *If $\Gamma\vdash\varphi:\mathbb{F}$ then we have an isomorphism of types*

$$\Gamma\vdash\triangleright\xi.\Pi(p:[\varphi]).B\cong\Pi(x:[\varphi]).\triangleright\xi.B.\tag{2}$$

Proof. Using Proposition 2.31 it suffices to show that $\Gamma\vdash[\varphi]$ is constant with respect to ω. Using Lemmas 2.29 and 2.30 it further suffices to show that the presheaf $\mathbb{F}$ is in the essential image of π^*, which is exactly what Lemma 2.19 states. □

2.5.2 Interpreting later types

Lemma 2.33. *Formation of $\triangleright\xi$-types preserves compositions. More precisely, if $\triangleright\xi.A$ is a well-formed type in context Γ and we have a composition term $\mathbf{c}_A:\Phi(\Gamma,\Gamma';A)$, then there is a composition term $\mathbf{c}:\Phi(\Gamma;\triangleright\xi.A)$.*

Note that the types in Γ' need not be fibrant.

Proof. We introduce the following variables:

$$\begin{aligned}\gamma&:\mathbb{I}\to\Gamma\\\varphi&:\mathbb{F}\\u&:\Pi(i:\mathbb{I}).\,((\triangleright\xi.A)(\gamma\,i))^{\varphi}\\a_0&:(\triangleright\xi.A)(\gamma\,0)[\varphi\mapsto u\,0].\end{aligned}$$

Using Lemma 2.27 we can rewrite the types of u and a_0:

$$\begin{aligned}u&:\Pi(i:\mathbb{I}).\left(\triangleright\left[\gamma'\leftarrow\overline{\xi\circ\gamma}\right].A(\gamma'\,i)\right)^{\varphi}\\a_0&:\triangleright\left[\gamma'\leftarrow\overline{\xi\circ\gamma}\right].A(\gamma'\,0).\end{aligned}$$

Furthermore, we have the following type isomorphisms:

$$\begin{aligned}\Pi(i:\mathbb{I}).\left(\triangleright\left[\gamma'\leftarrow\overline{\xi\circ\gamma}\right].A(\gamma'\,i)\right)^{\varphi} &\cong \Pi(i:\mathbb{I}).\triangleright\left[\gamma'\leftarrow\overline{\xi\circ\gamma}\right].\left(A(\gamma'\,i)\right)^{\varphi} && \text{(Corr. 2.32)}\\ &\cong \triangleright\left[\gamma'\leftarrow\overline{\xi\circ\gamma}\right].\Pi(i:\mathbb{I}).\left(A(\gamma'\,i)\right)^{\varphi}, && \text{(Prop. 2.31)}\end{aligned}$$

which means that we have a term

$$\tilde{u}:\triangleright\left[\gamma'\leftarrow\overline{\xi\circ\gamma}\right].\Pi(i:\mathbb{I}).\left(A(\gamma'\,i)\right)^{\varphi}.$$

We can now – almost – form the term

$$\mathsf{next}\begin{bmatrix}\gamma'\leftarrow\overline{\xi\circ\gamma}\\ u'\leftarrow\tilde{u}\\ a_0'\leftarrow a_0\end{bmatrix}.\mathbf{c}_A\,\gamma'\,\varphi\,u'\,a_0' \;:\; \triangleright\left[\gamma'\leftarrow\overline{\xi\circ\gamma}\right].A(\gamma'\,1). \qquad (*)$$

In order for the composition sub-term to be well-typed, we need that $a_0' = u\,0$ under the assumption φ. This is equivalent to saying that the type

$$\triangleright\begin{bmatrix}\gamma'\leftarrow\overline{\xi\circ\gamma}\\ u'\leftarrow\tilde{u}\\ a_0'\leftarrow a_0\end{bmatrix}.(\mathrm{Id}(a_0',u'\,0))^{\varphi}$$

is inhabited. We transform the type as follows:

$$\begin{aligned}\triangleright\begin{bmatrix}\gamma'\leftarrow\overline{\xi\circ\gamma}\\ u'\leftarrow\tilde{u}\\ a_0'\leftarrow a_0\end{bmatrix}.(\mathrm{Id}(a_0',u'\,0))^{\varphi} &\cong \left(\triangleright\begin{bmatrix}\gamma'\leftarrow\overline{\xi\circ\gamma}\\ u'\leftarrow\tilde{u}\\ a_0'\leftarrow a_0\end{bmatrix}.\mathrm{Id}(a_0',u'\,0)\right)^{\varphi} && \text{(Corr. 2.32)}\\ &= \left(\mathrm{Id}(\mathsf{next}\begin{bmatrix}\gamma'\leftarrow\overline{\xi\circ\gamma}\\ u'\leftarrow\tilde{u}\\ a_0'\leftarrow a_0\end{bmatrix}.a_0', \mathsf{next}\begin{bmatrix}\gamma'\leftarrow\overline{\xi\circ\gamma}\\ u'\leftarrow\tilde{u}\\ a_0'\leftarrow a_0\end{bmatrix}.u'\,0)\right)^{\varphi}\\ &= (\mathrm{Id}(a_0,u\,0))^{\varphi},\end{aligned}$$

where the last equality uses that $\tilde{u}$ is defined using the inverse of $\lambda f\lambda x.\,\mathsf{next}\,\xi\,[f'\leftarrow f]\,.f'\,x$ (Prop. 2.31). By assumption it is the case that $(\mathrm{Id}(a_0,u\,0))^{\varphi}$ is inhabited, and therefore $(*)$ is well-defined. This concludes the existence part proof, as

$$\triangleright\left[\gamma'\leftarrow\overline{\xi\circ\gamma}\right].A(\gamma'\,1) = (\triangleright\xi.A)(\gamma\,1),$$

by Lemma 2.27.

We now have to show that the $(*)$ is equal to $u\,1$ under the assumption of φ. Assuming φ, we get by the properties of $\mathbf{c}_A$ that

$$\mathsf{next}\begin{bmatrix}\gamma'\leftarrow\overline{\xi\circ\gamma}\\ u'\leftarrow\tilde{u}\\ a_0'\leftarrow a_0\end{bmatrix}.\mathbf{c}_A\,\gamma'\,\varphi\,u'\,a_0' = \mathsf{next}\begin{bmatrix}\gamma'\leftarrow\overline{\xi\circ\gamma}\\ u'\leftarrow\tilde{u}\\ a_0'\leftarrow a_0\end{bmatrix}.u'\,1,$$

and by the definition of $\tilde{u}$ (Prop. 2.31) we have that

$$\mathsf{next}\begin{bmatrix}\gamma'\leftarrow\overline{\xi\circ\gamma}\\ u'\leftarrow\tilde{u}\\ a_0'\leftarrow a_0\end{bmatrix}.u'\,1 = u\,1$$

as desired. □

2.6 Summary of the semantics of GCTT

The interpretation of the syntax of GCTT follows the usual pattern for interpreting dependent type theory, see, e.g., the handbook chapter [10]: we define a partial function on raw types and terms and then show that it is defined and independent of the derivation on all derivable judgements.

In all we define the interpretations of the following judgements with the help of the internal language of $\widehat{\mathcal{C} \times \omega}$.

- $[\![\Gamma \vdash]\!]$
- $[\![\Gamma \vdash A]\!]$
- $[\![\Gamma \vdash t : A]\!]$
- $[\![\Gamma \vdash A = B]\!]$
- $[\![\Gamma \vdash t = s : A]\!]$
- $[\![\vdash \xi : \Gamma \twoheadrightarrow \Gamma']\!]$
- $[\![\rho : \Gamma \to \Gamma']\!]$

where the last one is a context morphism.

In particular soundness of the interpretation states that if

$$[\![\Gamma \vdash A = B]\!]$$

then the types $[\![\Gamma \vdash A]\!]$ and $[\![\Gamma \vdash B]\!]$ are interpreted as the same object. We have an analogous result for the judgement

$$[\![\Gamma \vdash t = s : A]\!] .$$

and the interpretation of terms $[\![\Gamma \vdash t : A]\!]$ and $[\![\Gamma \vdash s : A]\!]$.

References

[1] Lars Birkedal, Rasmus Ejlers Møgelberg, Jan Schwinghammer, and Kristian Støvring. First steps in synthetic guarded domain theory: step-indexing in the topos of trees. *LMCS*, 8(4), 2012.

[2] Garrett Birkhoff et al. Rings of sets. *Duke Mathematical Journal*, 3(3):443–454, 1937.

[3] Cyril Cohen, Thierry Coquand, Simon Huber, and Anders Mörtberg. Cubical type theory: a constructive interpretation of the univalence axiom. Unpublished, 2016.

[4] William H Cornish and Peter R Fowler. Coproducts of de morgan algebras. *Bulletin of the Australian Mathematical Society*, 16(01):1–13, 1977.

[5] Peter Dybjer. Internal type theory. In *TYPES '95*, pages 120–134, 1996.

[6] Peter T. Johnstone. *Sketches of an Elephant: A Topos Theory Compendium.* OUP, 2002.

[7] Saunders Mac Lane. *Categories for the working mathematician*, volume 5. Springer Science & Business Media, 1978.

[8] Saunders Mac Lane and Ieke Moerdijk. *Sheaves in Geometry and Logic.* Springer, 1992.

[9] Wesley Phoa. An introduction to fibrations, topos theory, the effective topos and modest sets. Technical Report ECS-LFCS-92-208, LFCS, University of Edinburgh, 1992.

[10] A. M. Pitts. Categorical logic. In S. Abramsky, D. M. Gabbay, and T. S. E. Maibaum, editors, *Handbook of Logic in Computer Science, Volume 5. Algebraic and Logical Structures*, chapter 2, pages 39–128. Oxford University Press, 2000.